RANDOM WALKS ON VERTEX-TRANSITIVE GRAPHS WITH MODERATE GROWTH

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ABSTRACT. We study the mixing time of symmetric random walks on vertex-transitive graphs. Given a vertex-transitive graph Γ with diameter γ, Diaconis and Saloff-Coste showed that if Γ is a Cayley graph of moderate growth then the mixing time of the simple random walk on Γ is quadratic in γ. The main technique they used was bounding the return probability and the spectral gap of the walk. We will generalise their result to arbitrary finite vertex-transitive graphs.

CONTENTS

1. INTRODUCTION

Throughout this paper, we denote by Γ a finite, connected, undirected graph without multiple edges. We begin by introducing the definitions required for the statement of the main theorem. Recall that every random walk on Γ is associated with a stochastic matrix P which contains the transition probabilities of the walk. For $u, v \in V$, we denote by P_u the u-th row of P; this is a probability vector with non-negative entries that sum to one. The (u,v)-th entry of P is denoted by $P(u,v)$ or $P_u(v)$, and represents the probability of moving from u to v given that we are at u.

We define a *transition system* (Γ, P) to be a graph Γ equipped with a transition matrix P associated with a random walk on Γ. We say that (Γ, P) is *symmetric* if P is symmetric. We denote by U the matrix associated with the *uniform distribution*, i.e. $U(u,v) = \frac{1}{N}$ for all $u, v \in V$. We write $\mathrm{Aut}(\Gamma)$ for the group of graph automorphisms of Γ, acting on the left. We define the set of automorphisms of Γ *preserved* by P to be

$$\mathrm{Aut}(\Gamma, P) = \{\alpha \in \mathrm{Aut}(\Gamma) : \forall u, v \in V,\ P(u,v) = P(\alpha(u), \alpha(v))\}.$$

Observe that this is a subgroup of $\mathrm{Aut}(\Gamma)$. We say that the system (Γ, P) is *transitive* if $\mathrm{Aut}(\Gamma, P)$ acts transitively on Γ. In particular, if (Γ, P) is transitive, then Γ is vertex-transitive. Note that symmetry and transitivity of (Γ, P) do not imply one another.

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The main purpose of this paper is to generalise the result of Diaconis and Saloff-Coste in [1] on Cayley graphs with moderate growth to vertex-transitive graphs with moderate growth. Recall that, given constants A and d, we say a finite connected vertex-transitive graph Γ has (A,d)-*moderate growth* if

$$\beta(j) \geq \frac{1}{A}\left(\frac{j}{\gamma}\right)^{d}|\Gamma| \quad \text{whenever} \quad 1 \leq j \leq \gamma.$$

The main result of this paper is the following quadratic bound on the mixing time in ℓ^2.

Theorem 1.1. *Let (Γ, P) be a symmetric transitive system. Suppose Γ is finite, connected, and has (A,d)-moderate growth. Let $\gamma = \mathrm{diam}(\Gamma)$.*

(i) For an upper bound, suppose that $\eta = \inf\{P(u,v) : u \sim v\} > 0$ and $\delta = \inf\{P(u,u) : u \in V\} \geq \frac{\eta}{2\gamma^2}$. Then

$$\frac{\|(P^n - U)_o\|_2}{\|U_o\|_2} \leq Be^{-c},$$

where $B = A^{1/2}2^{1+\frac{d+d^2}{4}}$ and $c = \frac{n\eta}{\gamma^2} > 0$.

(ii) For a lower bound, suppose that $P(u,v) = 0$ for any two non-adjacent vertices $u, v \in V$, and that $\gamma \geq A2^{2d+2}$. Then

$$\|(P^n - U)_o\|_1 \geq e^{-c} \quad \text{for} \quad c = \frac{2^{4d+3}A^2 n}{\gamma^2}.$$

Our proof builds on the work of Diaconis and Saloff-Coste [3]. We streamline their arguments, strengthen the results, and provide further clarification where needed. Recall that a symmetric transition matrix P has real eigenvalues $1 = \pi_1 \geq \pi_2 \geq \cdots \geq \pi_N \geq -1$. Given such a matrix P, we set $\pi_* = \max\{|\pi_2|, |\pi_N|\}$, the second-largest eigenvalue of P in absolute value. The following result motivates our study of the return probability and the eigenvalues of the random walk, forming our strategy for proving Theorem 1.1.

Lemma 1.2. *Suppose (Γ, P) is a symmetric system. Let $\pi_* = \max\{|\pi_2|, |\pi_N|\}$. Then for non-negative integers n and m,*

$$\|(P^{n+m} - U)_o\|_2^2 \leq P^{2m}(o,o)\,\pi_*^{2n}. \tag{1}$$

The structure of this paper is as follows. In Section 2, we introduce further notation and standard results required for this paper. In Section 3, we prove Lemma 1.2. In Section 4, we bound the return probability $P^{2m}(o,o)$, and in Section 5, we bound the eigenvalue term π_*^{2n}. In Section 6, we combine these bounds to prove Theorem 1.1; in fact, we also deduce mixing time bounds in any ℓ^p space from Theorem 1.1. Finally, Tointon and Tessera proved that vertex-transitive graphs with a suitable large diameter condition have moderate growth [4]; we show in Section 7 that our results imply that such graphs with large diameter also have quadratic mixing time, addressing a remark raised in the paper of Tessera and Tointon [4, Remark 2.9].

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2. Preliminaries

With the setup of Section 1 in place, we introduce additional notation. Let Γ be a finite, connected, undirected graph without multiple edges. We write $V = V(\Gamma)$ for the vertex set of Γ. We often assume that Γ has $|V| = N$ vertices. For u, $v \in V$, we write $u \sim v$ if u and v are connected by an edge; we denote by $d_\Gamma(u,v)$ the number of edges in a shortest path from u to v. Taking a starting vertex $o \in V$, we denote $|v| = d_\Gamma(v,o)$ to be the number of edges in a shortest path from o to v. Given $j \in \mathbb{N}$, we write $B_\Gamma(v,j) = \{u \in V : d(u,v) \leq j\}$ to be the ball of radius j centred at v. Lastly, in the case when Γ is vertex-transitive, the size of the ball, $|B_\Gamma(v,j)|$ is the same for every $v \in V$; we write $\beta_\Gamma(j)$ for this cardinality. Also, we write $\gamma = \mathit{diam}(\Gamma) = \min\{j : \beta_\Gamma(j) = |V|\}$, the diameter of the graph. We might often drop the subscript Γ if it is clear from the context. Denote $\deg(\Gamma)$ to be the degree of every vertex in Γ. The matrix P associated with the *simple random walk* is defined as follows:

$$P(u,v) = \begin{cases} \frac{1}{\deg \Gamma + 1} & \text{if } u \sim v \text{ or } u = v, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

As we mentioned in Section 1, a system (Γ, P) is transitive if the probability of going from any vertex to another is invariant under automorphisms. By induction, it follows that the probability of going from one vertex to another one in a finite number of steps is also invariant under automorphisms.

Lemma 2.1. *Let (Γ, P) be a transition system, for $n \geq 1$, we have*

$$\mathrm{Aut}(\Gamma, P) \subseteq \mathrm{Aut}(\Gamma, P^n). \tag{3}$$

Therefore, if (Γ, P) is a transitive system, then for any $n \geq 1$, (Γ, P^n) is also a transitive system.

Proof. We will prove by induction on n. The statement is clear for $n = 1$. Suppose it holds for $n-1$, and let $\alpha \in \mathrm{Aut}(\Gamma, P) \subseteq \mathrm{Aut}(\Gamma, P^{n-1})$. Then for $u, v \in V$, we have

$$\begin{aligned} P^n(u,v) &= \sum_{w \in V} P^{n-1}(u,w)P(w,v) \\ &= \sum_{w \in V} P^{n-1}(\alpha(u),\alpha(w))P(\alpha(w),\alpha(v)) \\ &= \sum_{z \in V} P^{n-1}(\alpha(u),z)P(z,\alpha(v)) \qquad \text{since } \alpha \text{ is a bijection on } V \\ &= P^n(\alpha(u),\alpha(v)). \end{aligned}$$

Hence, $\mathrm{Aut}(\Gamma, P) \subseteq \mathrm{Aut}(\Gamma, P^n)$. □

Remark. We will give an example when the other inclusion in (3) does not hold. Take $\Gamma = K_3$, the complete graph on 3 vertices v_1, v_2 and v_3. Consider

$$P = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Then $\mathrm{Aut}(\Gamma, P^2) = \mathrm{Aut}(\Gamma, I_3) \cong S_3$. Let α be an automorphisms on Γ such that $\alpha(v_1) = v_2$. Then $P(v_1,v_1) = 1 \neq 0 = P(v_2,v_2) = P(\alpha(v_1),\alpha(v_1))$, so $\alpha \notin \mathrm{Aut}(\Gamma, P)$.

For a transitive system, the following result shows that the set of transition probabilities at a given vertex is the same for all vertices. In fact, the columns of the transition matrix have the same entries up to permutation by a graph automorphism.

Lemma 2.2. *Suppose* (Γ, P) *is a transitive system, then for* $u,\, v \in V$*, there exists* $\alpha \in \mathrm{Aut}(\Gamma, P)$*, such that* $P(u,w) = P(v, \alpha(w))$ *for all* $w \in V$*.*

Proof. Since (Γ, P) is transitive, there exists $\alpha \in \mathrm{Aut}(\Gamma, P)$, such that $\alpha(u) = v$. Hence, for all $w \in V$, we have $P(u,w) = P(\alpha(u), \alpha(w)) = P(v, \alpha(w))$. □

Next, we introduce some standard definitions and results from functional analysis. Any real-valued function f on the vertex set V of Γ, with $|V| = N$, can be viewed as a vector in $\mathbb{R}^N$. For $1 \le s \le t \le \infty$, we define the ℓ^s norm on $\mathbb{R}^N$ in the usual way:

$$\|f\|_s = \left(\sum_{u \in V} |f(u)|^s \right)^{1/s}, \qquad \|f\|_\infty = \max_{u \in V} |f(u)|.$$

Given a matrix $Q \in \mathbb{R}^{N \times N}$, we define the operator norm

$$\|Q\|_{s \to t} = \sup_{\|f\|_s = 1} \|Qf\|_t.$$

Recall that for $1 \le s \le t \le \infty$, we also have

$$\|f\|_s \le N^{\frac{1}{s} - \frac{1}{t}} \|f\|_t. \tag{4}$$

For real functions f, g on a finite set V, we define their *inner product* to be

$$\langle f, g \rangle = \sum_{v \in V} f(v) g(v).$$

Let P be a symmetric and stochastic matrix, the *Dirichlet form* associated with P on V is defined by

$$\mathscr{E}_P = \mathscr{E}_P(f,f) = \langle (I - P) f, f \rangle.$$

The following result gives us an expression of the Dirichlet form as a double sum over the vertices of V.

Lemma 2.3. *Let* P *be a symmetric and stochastic matrix, for any real function* f *on finite set* V*, we have*

$$\mathscr{E}_P(f,f) = \frac{1}{2} \sum_{u,v \in V} (f(u) - f(v))^2 P(u,v).$$

Proof. Since P is symmetric, for any real function g on V, we have

$$\sum_{u,v \in V} g(u) P(u,v) = \sum_{u,v \in V} g(u) P(v,u) = \sum_{u,v \in V} g(v) P(u,v). \tag{5}$$

Therefore,

$$\begin{aligned} \langle (I-P)f, f \rangle &= \sum_{u \in V} \big[(If - Pf)(u) \big] f(u) \\ &= \sum_{u \in V} \left[f(u) - \sum_{v \in V} P(u,v) f(v) \right] f(u) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{u\in V}\left[f(u)^2 - \sum_{v\in V} P(u,v)f(u)f(v)\right] \\
&= \sum_{u\in V}\sum_{v\in V}\left[f(u)^2P(u,v) - f(u)f(v)P(u,v)\right] \quad \text{since P is stochastic} \\
&= \frac{1}{2}\sum_{u,v\in V}\left[f(u)^2P(u,v) + f(u)^2P(u,v) - 2f(u)f(v)P(u,v)\right] \\
&= \frac{1}{2}\sum_{u,v\in V}\left[f(u)^2P(u,v) + f(v)^2P(u,v) - 2f(u)f(v)P(u,v)\right] \quad \text{by (5)} \\
&= \frac{1}{2}\sum_{u,v\in V}\Big(f(u) - f(v)\Big)^2 P(u,v).
\end{aligned}
$$

□

3. Bounding $\|(P^n - U)_o\|_2^2$ by the return probability and the eigenvalue

We begin this section by establishing equality between the norm of a symmetric operator in the l^2 space and its eigenvalues.

Lemma 3.1. *Let Q be a real symmetric $N \times N$ matrix with eigenvalues $\{\pi_i\}_{i=1}^N$, ordered so that $|\pi_1| = \max_i |\pi_i|$. Then*

$$\|Q\|_{2\to 2} = |\pi_1|.$$

Proof. Since Q is symmetric, it admits an orthonormal eigenbasis $\{f_i\}_{i=1}^N$, where $Qf_i = \pi_i f_i$ with $|\pi_1| = \max_i |\pi_i|$. For any $f = \sum_{i=1}^N a_i f_i$ with $\|f\|_2 = 1$, we have $\sum_{i=1}^N a_i^2 = 1$ and

$$\|Qf\|_2^2 = \sum_{i=1}^N a_i^2 \pi_i^2 \le |\pi_1|^2.$$

Hence $\|Q\|_{2\to 2} \le |\pi_1|$. Equality follows by taking $f = f_1$. □

Recall that a symmetric transition matrix P has real eigenvalues $1 = \pi_1 \ge \pi_2 \ge \cdots \ge \pi_N \ge -1$. The above lemma applies to our case as follows.

Lemma 3.2. *Let P be a real symmetric transition matrix. Let $1 = \pi_1 \ge \pi_2 \ge \cdots \ge \pi_N \ge -1$ be its eigenvalues, with corresponding orthogonal eigenvectors $f_1 = (1, \dots, 1), f_2, \dots, f_N$. Then:*

(i) $f_1, f_2, \dots, f_N$ are also eigenvectors of U, corresponding to the eigenvalues $1, 0, \dots, 0$, respectively.

(ii) $\|P - U\|_{2\to 2} = \pi_*$, where $\pi_* = \max\{|\pi_2|, |\pi_N|\}$.

Proof. (i) Since U is the uniform transition matrix, we have

$$Uf_i = \frac{1}{N}\big(\langle f_1, f_i\rangle, \dots, \langle f_1, f_i\rangle\big).$$

For $i \ge 2$, orthogonality gives $\langle f_1, f_i\rangle = 0$, hence $Uf_i = 0$. For $i = 1$, we have $Uf_1 = f_1$. Thus the eigenvalues of U are $1, 0, \dots, 0$.

(ii) From (i),
$$(P-U)f_i = \begin{cases} 0 & \text{if } i = 1, \\ \pi_i f_i & \text{if } i \geq 2. \end{cases}$$
Thus the eigenvalues of $P-U$ are $\pi_2, \dots, \pi_N$ and 0, so the largest absolute eigenvalue is π_*. The result follows from Lemma 3.1.

□

For a transitive system (Γ, P), Lemma 2.2 shows that the entries of any two rows are permutations of each other. The following result relates the operator norm to the row vectors of such matrices.

Lemma 3.3. *Let T be an $N \times N$ matrix with rows and columns indexed by a finite set V. Suppose that T satisfies*

$$\forall u, v \in V,\ \exists \sigma \in \operatorname{Aut}(V) \text{ such that } \forall x \in V,\ T_u(x) = T_v(\sigma(x)). \tag{6}$$

Let $1 \leq s, t \leq \infty$ with $\frac{1}{s} + \frac{1}{t} = 1$. Then, for any $o \in V$,
(i) $\|T\|_{s \to s} \leq \|T_o\|_1$;
(ii) $\|T\|_{s \to \infty} = \|T_o\|_t$.

It is clear that the transition matrix of a transitive system satisfies (6).

Proof. By the definition of the operator norm, we have
$$\begin{aligned}
\|T\|_{s\to\infty} &= \sup_{\|f\|_s = 1} \|Tf\|_\infty \\
&= \sup_{\|f\|_s=1} \max_{u\in V} \left| \sum_{y \in V} T_u(y) f(y) \right| && \text{using definition of matrix–vector multiplication} \\
&\leq \sup_{\|f\|_s=1} \max_{u\in V} \|T_u\|_t \|f\|_s && \text{by Hölder's inequality} \\
&= \max_{u \in V} \|T_u\|_t = \|T_o\|_t && \text{by property (6).}
\end{aligned}$$

This proves the $s = \infty$ case in (i) and $(\leq)$ direction in (ii).

(i) Suppose $s < \infty$, we have
$$\begin{aligned}
\|T\|_{s\to s} &= \sup_{\|f\|_s=1} \|Tf\|_s \\
&= \sup_{\|f\|_s=1} \left(\sum_{u\in V} \left| \sum_{y\in V} T_u(y) f(y) \right|^s \right)^{\frac{1}{s}} \\
&\leq \sup_{\|f\|_s=1} \left(\sum_{u\in V} \left(\sum_{y\in V} |T_u(y) f(y)| \right)^s \right)^{\frac{1}{s}} \\
&\leq \sup_{\|f\|_s=1} \left(\sum_{u\in V} (\|T_u\|_t \|f\|_s)^s \right)^{\frac{1}{s}} && \text{by Hölder's inequality} \\
&= N^{\frac{1}{s}} \|T_o\|_t && \text{by property (6)}
\end{aligned}$$

$$\le \|T_o\|_1.$$

where the last line follows from inequality (4).

(ii) It remains to prove ($\ge$) direction. Let $1 < s, t < \infty$. Define a real function h on V by

$$h(v) = |T_o(v)|^{t-1} \operatorname{sgn}(T_o(v)), \quad v \in V.$$

Then

$$\|h\|_s = \left(\sum_{v\in V} |T_o(v)^{(t-1)s}|\right)^{\frac{1}{s}} = \left(\sum_{v\in V} |T_o(v)|^t\right)^{\frac{1}{s}} = (\|T_o\|_t)^{\frac{t}{s}}.$$

Taking $f = \frac{h}{\|h\|_s}$, then $\|f\|_s = 1$ and we have

$$\begin{aligned}\|Tf\|_\infty &= \frac{1}{\|h\|_s} \max_{u\in V} \left|\sum_{v\in V} T_u(v)h(v)\right| \\ &\ge \frac{1}{\|h\|_s} \left|\sum_{v\in V} T_o(v)h(v)\right| \\ &= \frac{1}{\|h\|_s} \sum_{v\in V} |T_o(v)|^t \\ &= \frac{\|T_o\|_t^t}{\|h\|_s} = \|T_o\|_t.\end{aligned}$$

which proved the inequality. The cases $s = \infty$ or $t = \infty$ can be treated similarly.

□

We take note of these two special cases of the above lemma:

(i) If P is a stochastic matrix that satisfies property (6), then for $1 \le s \le \infty$,

$$\|P\|_{s\to s} \le 1 \tag{7}$$

(ii) When we take $s = t = 2$ in Lemma 3.3 (ii), we have

$$\|T\|_{2\to\infty} = \|T_o\|_2. \tag{8}$$

The following result is useful when we compute the difference between the n-step transition matrix and the uniform matrix.

Lemma 3.4. *(i) For any stochastic matrix P, we have $PU = UP = U$.*
(ii) Furthermore, for any $n \ge 1$, we have $P^n - U = (P - U)^n$.

Proof. (i) For any $u, v \in V$, we have

$$(PU)(u,v) = \sum_{w\in V} P(u,w)U(w,v) = \frac{1}{N}\sum_{w\in V} P(u,w) = \frac{1}{N}.$$

Similarly, we also have $(UP)(u,v) = \frac{1}{N}$.

(ii) We will prove by induction on n. The statement is clearly true for $n = 1$. Suppose it holds for $n - 1$, then

$$\begin{aligned}(P-U)^n &= (P-U)^{n-1}(P-U) \\ &= (P^{n-1} - U)(P-U) \qquad \text{by induction hypothesis}\end{aligned}$$

$$\begin{aligned} &= P^n - P^{n-1}U - UP + UU \\ &= P^n - U && \text{by part (i).} \end{aligned}$$

□

In the following lemma, we establish an equality between the return probability and the norm of the power of the transition matrix. Following from this, we also take a look at the decreasing pattern in the sequence $\{P^{2m}(o,o)\}_{m=1}^{\infty}$.

Lemma 3.5. *Suppose P is a symmetric stochastic matrix, then for any $m \geq 0$, we have*

(i) $P^{2m}(o,o) = \|P_o^m\|_2^2 = \|P_o^{2m}\|_\infty$;
(ii) $\|P_o^{m+1}\|_\infty \leq \|P_o^m\|_\infty$. *Therefore,* $P^{2m+2}(o,o) \leq P^{2m}(o,o)$.
(iii) $P^{2m+1}(o,o) \leq P^{2m}(o,o)$.

Proof. (i) Since P is symmetric,

$$P^{2m}(o,o) = \sum_{u \in V} P^m(o,u)P^m(u,o) = \sum_{u \in V} P_o^m(u)P_o^m(u) = \|P_o^m\|_2^2.$$

From this, we can see that $\|P_o^{2m}\|_\infty = \max_{v \in V} P_o^{2m}(v) \geq \|P_o^m\|_2^2$. To show the other direction of the inequality, we note that

$$\begin{aligned} \|P_o^{2m}\|_\infty &= \max_{v \in V} P_o^{2m}(v) \\ &= \max_{v \in V} \sum_{u \in V} P^m(o,u)P^m(u,v) \\ &\leq \max_{v \in V} \|P_o^m\|_2 \|P_v^m\|_2 \quad \text{by Cauchy Schwartz inequality} \\ &= \|P_o^m\|_2^2 \quad \text{by property (6).} \end{aligned}$$

(ii) By Hölder's inequality, we have

$$\|P_o^{m+1}\|_\infty = \max_{v \in \Gamma} \sum_{u \in \Gamma} P_o^m(u)P_u(v) \leq \max_{v \in \Gamma} \|P_o^m\|_\infty \|P_v\|_1 = \|P_o^m\|_\infty.$$

(iii) Again, by Hölder's inequality,

$$P^{2m+1}(o,o) = \sum_{y \in v} P_o^{2m}(y)P_o(y) \leq \left\|P_o^{2m}\right\|_\infty \|P_o\|_1 = P^{2m}(o,o).$$

□

Example 3.6. In general, we do not have $P^{m+1}(o,o) \leq P^m(o,o)$ for $m \geq 0$. For example, take $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we have $P^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

We can now give the proof for the result which we stated in the introduction.

Proof of Lemma 1.2. For any real function f on V, we have

$$\begin{aligned} \|(P^{n+m} - U)f\|_\infty &= \|P^m(P^n - U)f\|_\infty && \text{by Lemma 3.4} \\ &\leq \|P^m\|_{2\to\infty} \|P^n - U\|_{2\to 2} \|f\|_2. && (9) \end{aligned}$$

It follows that,

$$\|(P^{n+m} - U)_o\|_2^2 = \left\|P^{n+m} - U\right\|_{2\to\infty}^2 \qquad \text{by Lemma 2.2 and equality (8)}$$

$$
\begin{aligned}
&\leq \|P^m\|_{2\to\infty}^2 \|P^n - U\|_{2\to 2}^2 && \text{by (9)} \\
&= \|P_o^m\|_2^2 \|(P-U)^n\|_{2\to 2}^2 && \text{by equality (8) and Lemma 3.4} \\
&\leq P^{2m}(o,o)\pi_*^{2n} && \text{by Lemma 3.5 (i) and 3.2 (ii)}
\end{aligned}
$$

which completes the proof. □

4. The decay of the sequence $P^{2m}(o,o)$

We will be using the following standard results from linear algebra:

Lemma 4.1. *(i) Let P be a symmetric matrix. Then P is positive semi-definite if and only if all its eigenvalues are non-negative.*

(ii) A positive semi-definite matrix T has a unique positive semi-definite square root $T^{1/2}$. In particular, $T^{1/2}$ is symmetric.

(iii) A linear map on $\mathbb{R}^N$ defined by a symmetric matrix is self-adjoint.

When we have a semi-definite matrix T, we write $T^{1/2}$ to be its unique positive semi-definite square root.

Example 4.2. Denote I as the N by N identity matrix. Recall that the eigenvalues of a stochastic symmetric matrix P are bounded above by 1. Therefore, $I - P$ is positive semi-definite by Lemma 4.1 (i). Hence, by Lemma 4.1 (ii) and (iii), $(I-P)^{1/2}$ is self-adjoint.

Let (Γ, P) be a transitive system. We will write $P^+ = (I+P)/2$, which is still a stochastic matrix. We define $\eta = \eta_P = \inf\{P(u,v) : u \sim v\}$ and similarly $\eta^+ = \eta_P^+ = \inf\{P^+(u,v) : u \sim v\}$.

We now examine some properties of the variables associated with the matrix P^+.

Lemma 4.3. *Let (Γ, P) be a symmetric transitive system. Then*

(i) P^+ is positive semi-definite;

(ii) $\eta^+ = \frac{1}{2}\eta$;

(iii) For $m \geq 0$, we have $P^{2m}(o,o) \leq 2(P^+)^{2m}(o,o)$.

Proof. (i) Similar to Theorem 4.2, we know that the eigenvalues of P are bounded below by -1, so the eigenvalues of P^+ are bounded below by 0, the result follows from Lemma 4.1 (i).

(ii) This follows from the fact that the off-diagonal elements of I are 0, i.e. $I(u,v) = 0$ for $u \sim v$.

(iii) We have

$$
\begin{aligned}
(P^+)^{2m}(o,o) &= \left(\frac{I+P}{2}\right)^{2m}(o,o) \\
&= \frac{1}{2^{2m}} \sum_{j=0}^{2m} \binom{2m}{j} I^{2m-j} P^j(o,o) && \text{using Binomial expansion} \\
&\geq \frac{1}{2^{2m}} \sum_{j=0}^{m} \binom{2m}{2j} P^{2j}(o,o) \\
&\geq P^{2m}(o,o) \left(\frac{1}{2^{2m}} \sum_{j=0}^{m} \binom{2m}{2j}\right) && \text{by Lemma 3.5 (ii)}
\end{aligned}
$$

$$= \frac{P^{2m}(o,o)}{2}.$$

□

The proof of the next lemma requires the following observation from calculus. Given $m \in \mathbb{N}$, consider the function $h : [0,1] \to \mathbb{R}$, $h(x) = (1-x)x^{2m}$, by calculus, h obtains its maximum at $x = \frac{2m}{2m+1}$.

Recall a standard analysis result that the sequence $\left\{\left(\frac{n}{n+1}\right)^n\right\}_{n=1}^{\infty}$ is a positive decreasing sequence. Therefore,

$$\sup_{0 \le x \le 1} h(x) = \left(\frac{1}{2m+1}\right)\left(\frac{2m}{2m+1}\right)^{2m} \le \left(\frac{1}{2m+1}\right)\left(\frac{4}{9}\right) \le \frac{1}{4m}. \tag{10}$$

Lemma 4.4. *Suppose P is an N by N positive semi-definite stochastic matrix (hence symmetric). Then*

$$\|(I-P)^{\frac{1}{2}}P^m\|_{2\to 2} \le \frac{1}{2\sqrt{m}}.$$

Proof. Let $f_1, f_2, \dots f_N$ be a set of eigenvectors of P that form an orthonormal basis. Let π_i be the corresponding eigenvalues. Since P is positive semi-definite and stochastic, these eigenvalues are bounded between 0 and 1. Let f be a real function on $\mathbb{R}^N$, write $f = \sum_{i=1}^N a_i f_i$ with $a_i \in \mathbb{R}$, then

$$(I-P)^{\frac{1}{2}}P^m f = (I-P)^{\frac{1}{2}}\sum_{i=1}^N a_i f_i = (I-P)^{\frac{1}{2}}\sum_{i=1}^N a_i \pi_i f_i = \sum_{i=1}^N (1-\pi_i)^{\frac{1}{2}} a_i \pi_i f_i.$$

Using inequality (10), we obtain that

$$\left\|(I-P)^{\frac{1}{2}}P^m f\right\|_2^2 = \left\|\sum_{i=1}^N (1-\pi_i)^{\frac{1}{2}} a_i \pi_i f_i\right\|_2^2 \le \frac{1}{4m}\sum_{i=1}^N |a_i|^2 = \frac{1}{4m}\|f\|_2^2.$$

The result follows by taking the square root on both sides.

□

There is a very nice property about transitive action on a set, the following two sums are equal:

(1) the sum of a function over all the elements in the set;
(2) the sum of this function over all conjugacy classes, multiplied by the size of the stabiliser.

Lemma 4.5. *Let V be a finite set and H acts on V. Let f be a real function on V and let v be an element in V. Consider the following sums:*

(i) $S_1 = \sum_{\alpha \in H} f(\alpha(v))$
(ii) $S_2 = \sum_{u \in V} \sum_{\substack{\alpha \in H, \\ \alpha(v)=u}} f(\alpha(v))$
(iii) $S_3 = \sum_{u \in V} \sum_{\substack{\alpha \in H, \\ \alpha(u)=v}} f(\alpha(v))$.

Then $S_1 = S_2 = S_3$ for any group action H on V. In the case where H acts transitively on V, we have $|\mathrm{stab}(w)|$, the size of the stabiliser of w under H, is the same for every w in V, we denote by C to be this cardinality. Then the following sum also coincides with the above ones:

(iv) $S_4 = C\sum_{u \in V} f(u)$.

Proof. ($S_1 = S_2 = S_3$) This follows from the following observation:

$$\cup_{u\in V}\{\alpha \in H|\alpha(v) = u\} = H = \cup_{u\in V}\{\alpha \in H|\alpha(u) = v\}.$$

($S_2 = S_4$ when the action is transitive) Let $u \in V$, then there exists $\beta \in H$, such that $\beta(v) = u$. Also, $\{\alpha \in H|\alpha(v) = u\} = \beta\,\mathrm{stab}(v)$. Hence $|\{\alpha \in H|\alpha(v) = u\}| = |\,\mathrm{stab}(v)| = C$. □

We will see how the above lemma can be applied to the transitive system.

Lemma 4.6. *Let (Γ, P) be a transitive system, recall that this means that the group $\mathrm{Aut}(\Gamma, P)$ acts transitively on Γ. Let $x, u \in V$, such that $x \sim u$. Then*

$$\sum_{v\in V}\big(P_v(x) - P_v(u)\big)^2\eta \le \sum_{v\in V}\sum_{u\in V}\big(P_o(u) - P_o(v)\big)^2 P(u, v).$$

Proof. Let $C = |\,\mathrm{stab}(v)|$, the size of the stabiliser of v in $\mathrm{Aut}(\Gamma, P)$. We have

$$\begin{aligned}
&\sum_{v\in V}\Big[P_v(x) - P_v(u)\Big]^2\eta \\
&= \sum_{v\in V}\frac{1}{C}\sum_{\substack{\alpha\in\mathrm{Aut}(\Gamma,P),\\ \alpha(v)=o}}\Big[P_v(x) - P_v(u)\Big]^2\eta \\
&= \frac{1}{C}\sum_{v\in V}\sum_{\substack{\alpha\in\mathrm{Aut}(\Gamma,P),\\ \alpha(v)=o}}\Big[P_{\alpha(v)}(\alpha(x)) - P_{\alpha(v)}(\alpha(u))\Big]^2\eta \quad \text{since } (\Gamma, P) \text{ is transitive} \\
&= \frac{1}{C}\sum_{v\in V}\sum_{\substack{\alpha\in\mathrm{Aut}(\Gamma,P),\\ \alpha(v)=o}}\Big[P_o(\alpha(x)) - P_o(\alpha(u))\Big]^2\eta \\
&= \frac{1}{C}\sum_{\alpha\in\mathrm{Aut}(\Gamma,P)}\Big[P_o(\alpha(x)) - P_o(\alpha(u))\Big]^2\eta \qquad \text{by Lemma 4.5}(S_1 = S_3) \\
&\le \frac{1}{C}\sum_{\alpha\in\mathrm{Aut}(\Gamma,P)}\sum_{\substack{w\in V,\\ w\sim\alpha(x)}}\Big[P_o(\alpha(x)) - P_o(w)\Big]^2\eta \qquad \text{since } \alpha(x) \sim \alpha(u) \\
&\le \frac{1}{C}C\sum_{v\in V}\sum_{\substack{w\in V,\\ w\sim v}}\Big[P_o(v) - P_o(w)\Big]^2 P(v, w) \qquad \text{by Lemma 4.5}(S_1 = S_4) \\
&\le \sum_{v\in V}\sum_{u\in V}\big(P_o(u) - P_o(v)\big)^2 P(u, v).
\end{aligned}$$

□

The following lemma establishes a connection between the rate of decrease of the return probability P^{2m} and the volume growth of Γ.

Lemma 4.7. *Let (Γ, P) be a symmetric transitive system with P being a positive semi-definite matrix. Then, $\forall n, m \ge 0$'*

$$P^{2n+m}(o, o) \le \frac{2}{\beta(r(n, m))}, \quad \textit{where} \quad r(n, m) = \left(\frac{\eta}{2}\right)^{\frac{1}{2}} m^{\frac{1}{2}}\frac{P^{2n+m}(o, o)}{P^{2n}(o, o)}.$$

Proof. Let $x \in V$ and $u \sim x$. Let C be the stabiliser of a vertex under $\mathrm{Aut}(\Gamma, P)$. Then

$$\begin{aligned}
&|P_o^{2n+m}(x) - P_o^{2n+m}(u)| \\
&= \sum_{v \in V} \left| P_o^n(v) P_v^{n+m}(x) - P_o^n(v) P_v^{n+m}(u) \right| \\
&= \sum_{v \in V} \left| P_o^n(v) \right| \left| P_v^{n+m}(x) - P_v^{n+m}(u) \right| \\
&\le \left(\sum_{v \in V} \left| P_o^n(v) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{v \in V} \left(P_v^{n+m}(x) - P_v^{n+m}(u) \right)^2 \right)^{\frac{1}{2}} \text{ by Cauchy-Schwartz inequality} \\
&= \frac{\|P_o^n\|_2}{\eta^{\frac{1}{2}}} \left(\sum_{v \in V} \left(P_v^{n+m}(x) - P_v^{n+m}(u) \right)^2 \eta \right)^{\frac{1}{2}} \\
&\le \frac{\|P_o^n\|_2}{\eta^{\frac{1}{2}}} \left(\sum_{v \in V} \sum_{u \in V} \left(P_o^{n+m}(u) - P_o^{n+m}(v) \right)^2 P(u,v) \right)^{\frac{1}{2}} \quad \text{by Lemma 4.6} \\
&\le \frac{\|P_o^n\|_2}{\eta^{\frac{1}{2}}} \left(2 \langle (I-P) P_o^{n+m}, P_o^{n+m} \rangle \right)^{\frac{1}{2}} \quad \text{by Lemma 2.3} \\
&= \left(\frac{2}{\eta} \right)^{\frac{1}{2}} \|P_o^n\|_2 \left(\langle (I-P)^{\frac{1}{2}} P_o^{n+m}, (I-P)^{\frac{1}{2}} P_o^{n+m} \rangle \right)^{\frac{1}{2}} \quad \text{since } (I-P)^{\frac{1}{2}} \text{ is self-adjoint} \\
&= \left(\frac{2}{\eta} \right)^{\frac{1}{2}} \|P_o^n\|_2 \left\| (I-P)^{\frac{1}{2}} P^m P_o^n \right\|_2 \\
&\le \left(\frac{2}{\eta} \right)^{\frac{1}{2}} \|P_o^n\|_2 \|P_o^n\|_2 \left\| (I-P)^{\frac{1}{2}} P^m \right\|_{2 \to 2} \\
&\le \|P_o^n\|_2^2 \left(\frac{2}{\eta} \right)^{\frac{1}{2}} \frac{1}{2\sqrt{m}} \quad \text{by Lemma 4.4} \\
&\le \frac{\|P_o^n\|_2^2}{\sqrt{2m\eta}} = \frac{P^{2n}(o,o)}{\sqrt{2m\eta}} \quad \text{by Lemma 3.5 (i).}
\end{aligned}$$

Let $o = x_0, x_1, \ldots, x_{|x|} = x$ be a shortest path from o to x with $x_i \sim x_{i+1}$. The above calculation tells us that

$$|P_o^{2n+m}(x) - P_o^{2n+m}(o)| \le \sum_{i=0}^{|x|-1} |P_o^{2n+m}(x_{i+1}) - P_o^{2n+m}(x_i)| \le |x| \frac{P^{2n}(o,o)}{\sqrt{2m\eta}}.$$

Hence, for all $x \in V$, with $|x| \le r(n,m)$, we have

$$|P_o^{2n+m}(x) - P_o^{2n+m}(o)| \le r(n,m) \, \frac{P^{2n}(o,o)}{\sqrt{2m\eta}} = \left(\frac{\eta}{2} \right)^{\frac{1}{2}} m^{\frac{1}{2}} \frac{P^{2n+m}(o,o)}{P^{2n}(o,o)} \left(\frac{P^{2n}(o,o)}{\sqrt{2m\eta}} \right) \le \frac{P^{2n+m}(o,o)}{2}.$$

It follows that,

$$\frac{P^{2n+m}(o,o)}{2} \le P_o^{2n+m}(x).$$

Recall that there are $\beta(r(n,m))$ number of vertices which are at most $r(n,m)$ away from o, so summing the inequality over the set of $\{x \in V \big| |x| \leq r(n,m)\}$ gives us

$$\begin{aligned}\sum_{\substack{x\in V,\\|x|\leq r(n,m)}} \frac{P^{2n+m}(o,o)}{2} &\leq \sum_{\substack{x\in V,\\|x|\leq r(n,m)}} P_o^{2n+m}(x)\\ \beta(r(n,m))\frac{P^{2n+m}(o,o)}{2} &\leq 1\\ P^{2n+m}(o,o) &\leq \frac{2}{\beta(r(n,m))}\end{aligned}$$

which completes the proof. □

We will now find a bound for the decay of the sequence $P^{2m}(o,o)$ when the vertex-transitive graph undergoes polynomial growth.

Theorem 4.8. *Let (Γ, P) be a symmetric transitive system. Recall that we defined $\eta = \inf\{P(u,v) : u \sim v\}$. If the volume growth $\beta(j)$ satisfies*

$$\beta(j) \geq aj^d, \quad \text{for all} \quad 1 \leq j \leq M \tag{11}$$

for some positive a, d and M, then

$$P^{2m}(o,o) \leq \frac{D}{m^{d/2}} \quad \text{for all} \quad m \leq \frac{4M^2}{\eta}$$

with $D = 2^{2+\frac{3}{2}d+\frac{d^2}{2}}/(a\eta^{d/2})$.

Proof. Let us first assume that P is positive semi-definite. Set $A(n) = P^n(o,o)$. Then we have

$$\begin{aligned}A(2n+m) &\leq \frac{2}{\beta(r(n,m))} \quad \text{by Lemma 4.7}\\ &\leq \frac{2}{a} r(n,m)^{-d} \quad \text{using the volume bound (11)}\\ &= \frac{2}{a}\left\{\left(\frac{\eta}{2}\right)^{-1/2} m^{-1/2}\left(\frac{A(2n+m)}{A(2n)}\right)^{-1}\right\}^d,\end{aligned} \tag{12}$$

this holds for any n, m with $r(n,m) \leq M$. From the definition for r in Lemma 4.7 and the bounds in Lemma 3.5, $r(n,m) \leq \left(\frac{\eta m}{2}\right)^{\frac{1}{2}}$, therefore, the formulae (12) holds for $m \leq \frac{2M^2}{\eta}$ and for all n.

Rewriting (12) to make $A(2n+m)$ the subject, we get

$$A(2n+m) \leq \left\{\left(\frac{2}{a}\right)^{\frac{1}{d}}\left(\frac{2}{\eta}\right)^{\frac{1}{2}}\left(\frac{1}{m}\right)^{\frac{1}{2}} A(2n)\right\}^{\theta}, \tag{13}$$

where $\theta = \frac{d}{1+d}$. Fix $m_o \leq 4N^2\eta$. Let m be such that $2^m \leq m_0 < 2^{m+1}$. Then, by using (13) repeatedly, we have

$$A(2^m) = A(2^{m-1} + 2^{m-1})$$

$$\le \left\{ \left(\frac{2}{a}\right)^{\frac{1}{d}} \left(\frac{2}{\eta}\right)^{\frac{1}{2}} \left(\frac{1}{2^{m-1}}\right)^{\frac{1}{2}} A(2^{m-1}) \right\}^{\theta}$$
$$\le \left\{ \left(\frac{2}{a}\right)^{\frac{1}{d}} \left(\frac{2}{\eta}\right)^{\frac{1}{2}} \right\}^{\theta} \left\{ \frac{1}{2^{m-1}} \right\}^{\frac{\theta}{2}} \left\{ \left(\frac{2}{a}\right)^{\frac{1}{d}} \left(\frac{2}{\eta}\right)^{\frac{1}{2}} \left(\frac{1}{2^{m-2}}\right) A(2^{m-2}) \right\}^{\theta^2}$$
$$\vdots$$
$$\le \left\{ \left(\frac{2}{a}\right)^{\frac{1}{d}} \left(\frac{2}{\eta}\right)^{\frac{1}{2}} \right\}^{S_1(\theta)} \left\{ 2^{S_2(\theta)} \right\} A(2)^{\theta^{m-1}}$$

where

$$S_1(\theta) = \theta + \theta + \theta^2 \ldots + \theta^{m-1} = \sum_{i=1}^{m-1} \theta^i = d(1 - \theta^{m-1}) \le d;$$

and

$$S_2(\theta) = -\left[\frac{m-1}{2}\theta + \frac{m-2}{2}\theta^2 + \ldots \frac{1}{2}\theta^{m-1} \right] = \frac{1}{2} \left(\sum_{i=1}^{m-1} i\theta^i - m \sum_{i=1}^{m-1} \theta^i \right).$$

By differentiating $S_1(\theta)$ and multiplying the derivative by θ, we get that

$$\sum_{i=1}^{m-1} i\theta^i = (d + d^2)\big(1 - m\theta^{m-1} + (m-1)\theta^m\big).$$

We will use this to show that $S_2(\theta) \le \frac{d+d^2-md}{2}$ as follows:

$$\begin{aligned}
2S_2(\theta) - \Big(d + d^2 - md\Big) &= (d + d^2)\big(1 - m\theta^{m-1} + (m-1)\theta^m\big) - md(1 - \theta^{m-1}) \\
&\quad - \Big(d + d^2 - md\Big) \\
&= (d + d^2)\big[- m\theta^{m-1} + (m-1)\theta^m \big] + md\theta^{m-1} \\
&= -dm\theta^{m-1} + d(m-1)\theta^m - d^2 m\theta^{m-1} + d^2(m-1)\theta^m + md\theta^{m-1} \\
&= d(m-1)\theta^m - d^2 m\theta^{m-1} + d^2(m-1)\theta^m \\
&= \Big(d(m-1)\theta - d^2 m + d^2(m-1)\theta \Big)\theta^{m-1} \\
&= \Big((d+1)d(m-1)\theta - d^2 m \Big)\theta^{m-1} \\
&= \Big(d^2(m-1) - d^2 m \Big)\theta^{m-1} \qquad \text{since } \theta = \frac{d}{1+d} \\
&= -d^2\theta^{m-1} \le 0.
\end{aligned}$$

We obtain the following bound when P is positive semi-definite:

$$P^{m_0}(o, o) \le A(2^m) \qquad \text{by Lemma 3.5 (ii),}$$

$$
\begin{aligned}
&\leq \left\{ \left(\frac{2}{a}\right)^{\frac{1}{d}} \left(\frac{2}{\eta}\right)^{\frac{1}{2}} \right\}^{d} \left\{ 2^{\frac{d+d^2-md}{2}} \right\} \\
&= \left(\frac{2}{a}\right) \left(\frac{2}{\eta}\right)^{\frac{d}{2}} 2^{\frac{d+d^2}{2}} \frac{1}{2^{\frac{md}{2}}} \frac{2^{\frac{d}{2}}}{2^{\frac{d}{2}}} \\
&= \frac{2^{1+\frac{d}{2}+\frac{d+d^2}{2}+\frac{d}{2}}}{a\eta^{\frac{d}{2}}} \frac{1}{2^{\frac{(m+1)d}{2}}} \\
&\leq \frac{2^{1+(3d+d^2)/2}}{a(\eta m_0)^{\frac{d}{2}}} \qquad \text{since } 2^{m+1} \geq m_0.
\end{aligned}
$$

The above bound holds for the case when P is a positive semi-definite. For general P, Lemma 4.3 tells us that

$$
\begin{aligned}
P^{2m}(o,o) &\leq 2(P^{+})^{2m}(o,o) \\
&\leq 2\frac{2^{1+(3d+d^2)/2}}{a\big((\eta/2)2m\big)^{\frac{d}{2}}} \qquad \text{for } 2m \leq \frac{4M^2}{\eta/2} \\
&\leq \frac{2^{2+(3d+d^2)/2}}{a(\eta m)^{\frac{d}{2}}} \qquad \text{for } m \leq \frac{4M^2}{\eta}
\end{aligned}
$$

which completes the proof.

□

5. Bounds for the eigenvalues

In this section, we will try to find two quantitative bounds for eigenvalues π_2 and π_N.

5.1. **Lower bound for the smallest eigenvalue.**

Lemma 5.1. *Let P be a symmetric stochastic matrix with rows and columns indexed by elements in V. Let $1 = \pi_1 \geq \pi_2 \geq \ldots \geq \pi_N$ be the eigenvalues of P. Let $\delta = \inf\{P(u,u) | u \in V\}$, the minimum value of the diagonal elements of P. Then, the smallest eigenvalue of P satisfies*

$$\pi_N \geq -1 + 2\delta.$$

Proof. If $\delta = 1$, then P is the identity matrix with every eigenvalue being 1, so the result clearly holds. If $\delta < 1$, let $Q = (1-\delta)^{-1}(P - \delta I)$. Then Q is also a symmetric and stochastic matrix. Hence the eigenvalues of Q are bounded below by -1. The eigenvalues of Q are $(1-\delta)^{-1}(\pi_i - \delta)$, for $1 \leq i \leq N$. In particular, we have

$$(1-\delta)^{-1}(\pi_N - \delta) \geq -1$$

which gives us the result. □

5.2. **Lower bound and upper bound for the second biggest eigenvalue.**

By looking at the ratio of Dirichlet forms, we obtain the following upper and lower bounds for the second biggest eigenvalue.

Lemma 5.2. *Let Γ be a finite graph with order N. Let P be the matrix associated with a symmetric random walk, with eigenvalue $1 = \pi_1 \geq \pi_2 \geq \ldots \geq \pi_N$. Then,* *(i) Suppose* $\mathscr{E}_U \leq A\mathscr{E}_P$ *for some constant* A*, then* $\pi_i \leq 1 - 1/A$*, for all* $i \geq 2$.

(ii) For any non-zero real function g *on* V*, we have*

$$1 - \pi_2 \leq \frac{\mathscr{E}_P(g,g)}{\mathscr{E}_U(g,g)}.$$

Proof. Let $f_1, f_2, \dots, f_N$ be a set of orthonormal eigenvectors of P corresponding to eigenvalue $\pi_1 = 1, \pi_2 \dots, \pi_N$. By Lemma 3.2 (i), we see that $f_1, f_2, \dots, f_N$ are eigenvectors of U corresponding to eigenvalues $\tilde{\pi}_1 = 1, \tilde{\pi}_2 = 0, \tilde{\pi}_3 = 0, \dots, \tilde{\pi}_N = 0$.

(i) Suppose $\mathscr{E}_U \leq A\mathscr{E}_P$ for some constant A. Given $i \geq 2$, using Lemma 2.3, we have

$$A \geq \frac{\mathscr{E}_U(f_i,f_i)}{\mathscr{E}_P(f_i,f_i)} = \frac{\langle (I-U)f_i, f_i\rangle}{\langle (I-P)f_i, f_i\rangle} = \frac{\langle (1-\tilde{\pi}_i)f_i, f_i\rangle}{\langle (1-\pi_i)f_i, f_i\rangle} = \frac{1}{1-\pi_i}.$$

(ii) Let g be a non-zero function on V, with $g = \sum_{i=1}^N a_i f_i$. Again, using Lemma 2.3, we have

$$\begin{aligned}
\frac{\mathscr{E}_P(g,g)}{\mathscr{E}_U(g,g)} &= \frac{\langle (I-P)\sum_{i=1}^N a_i f_i, \sum_{i=1}^N a_i f_i\rangle}{\langle (I-U)\sum_{i=1}^N a_i f_i, \sum_{i=1}^N a_i f_i\rangle} \\
&= \frac{\langle \sum_{i=2}^N (1-\pi_i) a_i f_i, \sum_{i=2}^N a_i f_i\rangle}{\langle \sum_{i=2}^N a_i f_i, \sum_{i=2}^N a_i f_i\rangle} \quad \text{since } \pi_1 = \tilde{\pi}_1 = 1, \tilde{\pi}_2 = \tilde{\pi}_3 \dots \tilde{\pi}_N = 0, \\
&= \frac{\sum_{i=2}^N (1-\pi_i) a_i^2}{\sum_{i=2}^N a_i^2} \quad \text{since } f_2, f_3, \dots, f_N \text{ are orthonormal}, \\
&\geq \frac{\sum_{i=2}^N (1-\pi_2) a_i^2}{\sum_{i=2}^N a_i^2} \quad \text{since } \pi_2 \geq \pi_3 \geq \dots \geq \pi_N, \\
&= 1 - \pi_2.
\end{aligned}$$

□

We will now find a bound for the ratio of Dirichlet form on a vertex-transitive graph.

Theorem 5.3. *Let* Γ *be a finite connected vertex-transitive graph with* $\operatorname{diam}(\Gamma) = \gamma$*. Let* P *be the transition matrix for a random walk with* $\eta = \inf\{P(u,v) : u \sim v\} > 0$ *. Then we have*

$$\mathscr{E}_U \leq \frac{\gamma^2}{\eta}\mathscr{E}_P.$$

Proof. Since $\operatorname{Aut}(\Gamma)$ acts transitively on V, the size of the stabiliser of every v in V under the action $\operatorname{Aut}(\Gamma)$ is the same, we denote C to be this constant.

Take two vertices x, y in V, suppose $x_0 = x$, x_1, x_2, $\dots$, $x_k = y$ is a shortest path from x to y. Given $\alpha \in \operatorname{Aut}(\Gamma)$, note that $\alpha(x_0)$, $\alpha(x_1)$, $\alpha(x_2)$, $\dots$, $\alpha(x_k)$ is still a path, i.e. $\alpha(x_i) \sim \alpha(x_{i+1})$ for all i.

For any $\alpha \in \operatorname{Aut}(\Gamma)$, we have

$$\begin{aligned}
f(\alpha(x)) - f(\alpha(y)) = &\big[f(\alpha(x_0)) - f(\alpha(x_1))\big] + \\
&\big[f(\alpha(x_1)) - f(\alpha(x_2))\big] + \\
&\dots \\
&+ \big[f(\alpha(x_{k-1})) - f(\alpha(x_k))\big].
\end{aligned}$$

Then, squaring both sides and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\big[f(\alpha(x))-f(\alpha(y))\big]^2 \le |k|\Big\{&\big[f(\alpha(x_0))-f(\alpha(x_1))\big]^2+\\ &\big[f(\alpha(x_1))-f(\alpha(x_2))\big]^2+\\ &\dots\\ &+\big[f(\alpha(x_{k-1}))-f(\alpha(x_k))\big]^2\Big\}.\end{aligned} \tag{14}$$

We will do some manipulations on both sides of the inequality (14).

(**Step 1**)We will firstly sum over all $\alpha \in \mathrm{Aut}(\Gamma)$. On the left-hand side, we have

$$\sum_{\alpha\in\mathrm{Aut}(\Gamma)}\big[f(\alpha(x))-f(\alpha(y))\big]^2=\sum_{u\in V}\sum_{\substack{\alpha\in\mathrm{Aut}(\Gamma),\\ \alpha(x)=u}}\big[f(u)-f(\alpha(y))\big]^2$$

On the right-hand side, we have

$$\begin{aligned}\sum_{\alpha\in\mathrm{Aut}(\Gamma)}|k|\Big\{\sum_{i=0}^{k-1}\big[f(\alpha(x_i))-f(\alpha(x_{i+1}))\big]^2\Big\}&=|k|\sum_{i=0}^{k-1}\sum_{u\in V}\sum_{\substack{\alpha\in\mathrm{Aut}(\Gamma),\\ \alpha(x_i)=u}}\big[f(u)-f(\alpha(x_{i+1}))\big]^2\\ &\le |k|\sum_{i=0}^{k-1}\sum_{u\in V}\sum_{\substack{\alpha\in\mathrm{Aut}(\Gamma),\\ \alpha(x_i)=u}}\sum_{\substack{v\in V,\\ v\sim u}}\big[f(u)-f(v)\big]^2\\ &=|k|^2C\sum_{u\in V}\sum_{\substack{v\in V,\\ v\sim u}}\big[f(u)-f(v)\big]^2\\ &\le \gamma^2C\sum_{u\in V}\sum_{\substack{v\in V,\\ v\sim u}}\big[f(u)-f(v)\big]^2.\end{aligned}$$

(**Step 2**)Then we multiply both side by $U(x,y)=\frac{1}{|V|}$ and sum over $y\in\Gamma$. On the left-hand side, we have

$$\begin{aligned}&\sum_{y\in V}\sum_{u\in V}\sum_{\substack{\alpha\in\mathrm{Aut}(\Gamma),\\ \alpha(x)=u}}\big[f(u)-f(\alpha(y))\big]^2\frac{1}{|V|}\\ &=\sum_{u\in V}\sum_{\substack{\alpha\in\mathrm{Aut}(\Gamma),\\ \alpha(x)=u}}\sum_{y\in V}\big[f(u)-f(\alpha(y))\big]^2\frac{1}{|V|}\\ &=\sum_{u\in V}\sum_{\substack{\alpha\in\mathrm{Aut}(\Gamma),\\ \alpha(x)=u}}\sum_{v\in V}\big[f(u)-f(v)\big]^2\frac{1}{|V|} \qquad \text{since } \alpha \text{ is an automorphism}\\ &=C\sum_{u\in V}\sum_{v\in V}\big[f(u)-f(v)\big]^2\frac{1}{|V|}.\end{aligned}$$

Note that there are no y terms on the right-hand sides, multiplying by $\frac{1}{|V|}$ and summing over $y \in \Gamma$ have no effect:

$$\gamma^2 C \sum_{y\in V} \sum_{u\in V} \sum_{\substack{v\in V,\\ v\sim u}} \big[f(u)-f(v)\big]^2 \frac{1}{|V|} = \gamma^2 C \sum_{u\in V} \sum_{\substack{v\in V,\\ v\sim u}} \big[f(u)-f(v)\big]^2$$
$$\leq \gamma^2 C \frac{1}{\eta} \sum_{u\in V} \sum_{\substack{v\in V,\\ v\sim u}} \big[f(u)-f(v)\big]^2 P(u,v).$$

(**Step 3**)Finally, we divide both sides by $2C$, the left-hand side becomes $\mathscr{E}_U$, while the right-hand side gives us

$$\gamma^2 \frac{1}{\eta}\left\{\frac{1}{2}\sum_{u\in V}\sum_{\substack{v\in V\\ v\sim u}}[f(u)-f(v)]^2 P(u,v)\right\} \leq \frac{1}{\eta}\gamma^2\left\{\frac{1}{2}\sum_{u\in V}\sum_{v\in V}[f(u)-f(v)]^2 P(u,v)\right\} \leq \frac{\gamma^2}{\eta}\mathscr{E}_P.$$

□

Corollary 5.4. *Let Γ be a finite connected vertex-transitive graph. Let P be the matrix associated with a symmetric random walk. Suppose $\eta = \inf\{P(u,v) : u \sim v\} > 0$. Then the second biggest eigenvalue satisfies*

$$\pi_2 \leq 1 - \frac{\eta}{\gamma^2}.$$

Proof. This follows from Lemma 5.2 (i) and Theorem 5.3. □

6. Mixing time under Moderate growth

6.1. **Proof of the main theorem.** We begin with some observations on the moderate growth property defined in Section 1. Clearly, any vertex-transitive graph Γ has (A,d) moderate growth for some constant A and d, in fact, one can just take $A = |\Gamma|$. However, it turns out that many natural families vertex-transitive graphs have moderate growth with the same constant A and d.

Example 6.1. (1) For $m \in \mathbb{N}$, consider the Cayley graph Cayley$(\mathbb{Z}_m, \{-1,0,1\})$. We have $B(j) = 2j+1$ and $\operatorname{diam}(\mathcal{G}) = m/2$. It follows that Cayley$(\mathbb{Z}_m, \{-1,0,1\})$ has $(1,1)$-moderate growth.
(2) The Cayley graphs of finite nilpotent groups have (A,d)-moderate growth, where A and d depend only on the number of generators and the nilpotency class. This follows from [3, Chapter 5].
(3) In the same paper [3, Chapter 7], Diaconis and Saloff-Coste showed that the Cayley graphs of finite affine groups over $\mathbb{F}_p$ exhibit $(1,2)$-moderate growth, even though these groups have exponential growth in the sense that their size grows exponentially with the diameter of the graph.

Theorem 1.1 tells us that, if the graph undergoes moderate growth, by putting some restrictions when travelling between adjacency vertices, we can study the rate of convergence of the symmetric random walk to a uniform distribution. We present the proof here:

Proof of Theorem 1.1.

(i) Combining Lemma 5.1 and Corollary 5.4, we obtain that $\pi_* \le 1 - \frac{\eta}{\gamma^2}$. Since Γ has moderate growth, $\beta(j) \ge \frac{N}{A\gamma^d} j^d$ for $1 \le j \le \gamma$. Substitute $M = \gamma$, $a = \frac{N}{A\gamma^d}$ in Theorem 4.8 and take $m = \frac{4\gamma^2}{\eta}$, we obtain that

$$P^{2m}(o,o) \le \frac{2^{2+\frac{3}{2}d+\frac{d^2}{2}}}{\left(\frac{N}{A\gamma^d}\right)\left(\eta\frac{4\gamma^2}{\eta}\right)^{d/2}} = 2^{2+\frac{d+d^2}{2}}\frac{A}{N}.$$

Recall an analysis result that the function $h : \mathbb{R}_{\ge 1} \to \mathbb{R}$, $h(x) = (1 - \frac{1}{x})^x$ is an increasing function with $\lim_{x\to\infty} h(x) = e^{-1}$. Taking $n = c\gamma^2/\eta$, then $c = \frac{n\eta}{\gamma^2} = \frac{(m+n)\eta}{\gamma^2} - 4$ and then the bound (1) becomes

$$\begin{aligned} \|(P^{n+m} - U)_o\|_2^2 &\le 2^{2+\frac{d+d^2}{2}}\frac{A}{N}\left(1 - \frac{\eta}{\gamma^2}\right)^{2c\gamma^2/\eta} \\ &= 2^{2+\frac{d+d^2}{2}}\frac{A}{N}\left[\left(1 - \frac{\eta}{\gamma^2}\right)^{\gamma^2/\eta}\right]^{2c} \\ &\le 2^{2+\frac{d+d^2}{2}}\frac{A}{N}e^{-2c}. \end{aligned}$$

Finally, taking the square root on both sides and noting that $\|U_o\|_2 = N^{-\frac{1}{2}}$ gives the desired result.

(ii) Using Lemma 3.3 (ii), we have

$$\|(P^k - U)_o\|_1 = \|P^k - U\|_{\infty\to\infty} = \max_{\|f\|_\infty \le 1} \|P^k(f) - U(f)\|_\infty. \tag{15}$$

Let f be a real normalised eigenfunction for P corresponding to π_2 with respect to infinity norm, i.e. $Pf = \pi_2 f$ and $\|f\|_\infty = 1$. From Lemma 3.2 (i), we know that $U(f) = 0$. Hence, (15) tells us that

$$\|(P^k - U)_o\|_1 \ge \|P^k(f)\|_\infty = \pi_2^k. \tag{16}$$

Next, we will find the lower bound for π_2. We will show that

$$\pi_2 \ge 1 - \frac{D}{\gamma^2} \quad \text{with} \quad D = 4^{2d+1}A^2. \tag{17}$$

Let g be the graph distance function on V, i.e. $g(v) = |v|$ for $v \in V$. Lemma 5.2 (ii) tells us that

$$1 - \pi_2 \le \frac{\mathscr{E}_P(g,g)}{\mathscr{E}_U(g,g)}.$$

Recall that $\mathscr{E}_P(g,g) = \frac{1}{2}\sum_{u,v\in V}(g(u) - g(v))^2 P(u,v)$. We assumed that $P(u,v) = 0$ for any two non-adjacent vertices u, v in V, it follows that $(g(u) - g(v))^2 \le 1$ for $P(u,v) > 0$. Hence, $\mathscr{E}_P(g,g) \le \frac{1}{2}\sum_{u,v\in V} 1P(u,v) = \frac{N}{2}$. On the other hand,

$$\mathscr{E}_U(g,g) = \frac{1}{2}\sum_{u,v\in V}(g(u) - g(v))^2 U(u,v) = \frac{1}{2N}\sum_{0\le i,j\le\gamma}(i-j)^2 \partial B(i)\partial B(j),$$

where we write $\partial B(i) = |\{v \in V : |v| = i\}|$, the number of vertices that are distance i away from the o. Consider $B_1 = \{u \in V : |u| \le \frac{\gamma}{4}\}$, and $B_2 = \{v \in V : \frac{3\gamma}{4} \le |v| \le \gamma\}$, then for any $u \in B_1$, and $v \in B_2$, we have $\big||u| - |v|\big| \ge \frac{\gamma}{2}$. Therefore,

$$\mathscr{E}_U(g,g) \ge \frac{1}{2}\frac{\gamma^2}{4}|B_1||B_2|.$$

The moderate growth implies that

$$|B_1| = \beta(\gamma/4) \ge \frac{(\gamma/4)^d N}{\gamma^d A} = A^{-1}4^{-d}N.$$

Let $w \in V$ with $|w| = \gamma$. By triangle inequality,

$$z \in B(w, \frac{\gamma}{4}) \Rightarrow d(o,z) \ge d(o,w) - d(w,z) \ge \frac{3\gamma}{4} \Rightarrow z \in B_2,$$

so $B(w,\gamma/4) \subseteq B_2$. Hence, $|B_2| \ge |B(w,\gamma/4)| = \beta(\gamma/4) \ge A^{-1}4^{-d}N$. Combining these bounds we proved our claim (17), then the inequality (16) becomes

$$\|(P^n - U)_o\|_1 \ge \left(1 - \frac{A^2 4^{2d+1}}{\gamma^2}\right)^k.$$

The assumption $\gamma \ge A2^{2d+2}$ implies that $\frac{A^2 4^{2d+1}}{\gamma^2} \le \frac{1}{4}$. Furthermore, by looking at the Taylor expansion, we see that $1 - x \ge e^{-2x}$ for $0 \le x \le 1/2$, which concludes the proof.

□

6.2. **Mixing time.** For $\epsilon > 0$ and $1 \le p \le \infty$, the *l^p mixing time* of system (Γ, P) is

$$\tau_p(\epsilon) = \min\{n \ge 0 : \|(P^n - U)_o\|_p \le \epsilon \|U_o\|_p\}.$$

Following from Theorem1.1, we can deduce an upper bound and a lower bound for the mixing time for the random walk in the vertex-transitive graph.

Corollary 6.2. *(i) The mixing time for the system (Γ, P) described in Theorem 1.1 (i) is bounded above by*

$$\tau_2(\epsilon) \le \frac{\gamma^2}{\eta}\log\frac{A^{\frac{1}{2}}2^{1+\frac{d+d^2}{4}}e}{\epsilon}.$$

(ii) The mixing time for the system (Γ, P) described in Theorem 1.1 (ii) is bounded below by

$$\tau_1(\epsilon) \ge \frac{\gamma^2}{2^{4d+3}A^2}\log\frac{1}{\epsilon}.$$

We will use the following proposition to find an upper bound for the l^∞ mixing time.

Lemma 6.3. *(i) The quantity $\|(P^n - U)_o\|_p$ is non-increasing in n.*
(ii) The mixing time $\tau_p(\epsilon)$ is non-decreasing in p.
(iii) Suppose P is a symmetric stochastic matrix, then

$$\frac{\|(P^{2n} - U)_o\|_\infty}{\|U_o\|_\infty} = \left(\frac{\|(P^n - U)_o\|_2}{\|U_o\|_2}\right)^2.$$

(iv) Hence, $\tau_\infty(\epsilon) = 2\tau_2(\sqrt{\epsilon})$.

Proof. (i) According to inequality (7), we have

$$\|(P^{n+1}-U)_o\|_p = \|(P^n-U)_oP\|_p \le \|(P^n-U)_o\|_p\|P\|_{p\to p} \le \|(P^n-U)_o\|_p.$$

(ii) We begin by considering the case $1 \le p \le p' < \infty$. We know that the function $h : \mathbb{R}_{\ge 0} \to \mathbb{R}$, $h(x) = x^{p'/p}$ is a convex function. By Jensen's inequality, we have for any function $f : V \to \mathbb{R}$,

$$\left(\sum_{u\in V}\frac{1}{N}f(u)\right)^{\frac{p'}{p}} \le \sum_{u\in V}\frac{1}{N}f(u)^{\frac{p'}{p}}$$

$$\frac{\left(\sum_{u\in V}f(u)\right)^{\frac{p'}{p}}}{N^{\frac{p'}{p}}} \le \frac{\sum_{u\in V}f(u)^{\frac{p'}{p}}}{N}$$

$$\frac{\left(\sum_{u\in V}f(u)\right)^{\frac{1}{p}}}{N^{\frac{1}{p}}} \le \frac{\left(\sum_{u\in V}f(u)\right)^{\frac{p'}{p}})^{\frac{1}{p'}}}{N^{\frac{1}{p'}}}$$

This case is proven by taking $f(u) = |(P^n-U)(o,u)|^p$.

It remains to consider the case $p' = \infty$. By inequality (9), we have

$$\|(P^n-U)_o\|_p \le N^{\frac{1}{p}}\|(P^n-U)_o\|_\infty,$$

from which the conclusion follows.

(iii) For the numerator, Lemma 3.4 and Lemma 3.5 (i) tells us that

$$\|(P^{2n}-U)_o\|_\infty = \|(P-U)^{2n}_o\|_\infty = \|(P-U)^n_o\|_2^2 = \|(P^n-U)_o\|_2^2.$$

For the denominator, observe that $\|U_o\|_\infty = 1/N = \|U_o\|_2^2$.

(iv) We have

$$\begin{aligned}\tau_p(\epsilon) &= \min\{n \ge 0 : \frac{\|(P^n-U)_o\|_\infty}{\|U_o\|_\infty} \le \epsilon\}\\ &= \min\{n \ge 0 : \frac{\|(P^{\frac{n}{2}}-U)_o\|_2}{\|U_o\|_2} \le \sqrt{\epsilon}\}\\ &= \min\{2n \ge 0 : \frac{\|(P^n-U)_o\|_2}{\|U_o\|_2} \le \sqrt{\epsilon}\} = 2\tau_2(\sqrt{\epsilon}).\end{aligned}$$

□

Following from the above lemma, we can generalise our bounds in Example 6.2 to the l^p mixing time for $1 \le p \le \infty$,

Corollary 6.4. *(i) The mixing time for the system* (Γ, P) *described in Theorem 1.1 (i) is bounded above by*

$$\tau_p(\epsilon) \le \tau_\infty(\epsilon) \le \frac{\gamma^2}{\eta}\log\frac{A2^{2+\frac{d+d^2}{2}}e^2}{\epsilon}.$$

(ii) The mixing time for the system (Γ, P) *described in Theorem 1.1 (ii) is bounded below by*

$$\tau_p(\epsilon) \ge \tau_1(\epsilon) \ge \frac{\gamma^2}{2^{4d+3}A^2}\log\frac{1}{\epsilon}.$$

Remark. Suppose P is the transition matrix for the simple random walk defined in (2), the following authors have also worked on the mixing time for finite vertex transitive graph:

(i) In [4, Chapter 11.1], Hermon and Pymar pointed out that the lower bound of the l^∞ mixing time in Diaconis and Saloff-Coste's result would generalise to the vertex-transitive graph. They also proved an upper bound in [4, Proposition 11.1], while their bound depends quadratically on the degree of the graph.

(ii) In [3], Goel, Montenegro and Tetali proved that the l^∞ mixing time is comparable to γ^2, one can combine [3, Corollary 2.8] and a generalised version of [2, Lemma 5.3] to obtain an explicit upper bound for the l^∞ mixing time of symmetric random walks on vertex-transitive graphs.

7. Application: Vertex-transitive graph with large diameter

We will use the following asymptotic notation: for $X,Y \in \mathbb{R}$, we say $X = \mathcal{O}(Y)$ if X is at most a constant multiple of Y. If this constant depends on some other variable δ, we indicate this with a subscript and write $X = \mathcal{O}_\delta(Y)$. We state the following result from [4, Corollary 2.8], which tells us that a vertex-transitive graph with large diameter has moderate growth.

Theorem 7.1 (large diameter implies moderate growth)**.** *Let Γ be a finite connected vertex-transitive graph. For every $\delta \geq 0$, there exists $n_0 = n_0(\delta)$ such that if* $\operatorname{diam}(\Gamma) \geq n_0$ *and*

$$\operatorname{diam}(\Gamma) \geq \left(\frac{|\Gamma|}{\beta(1)}\right)^\delta \tag{18}$$

then Γ has $(\mathcal{O}_\delta(1), \mathcal{O}_\delta(1))$ - moderate growth.

Corollary 7.2 (large diameter implies quadratic mixing time)**.** *Let $\delta \geq 0$. Let (Γ, P) be a transition system, such that Γ also satisfies condition (18), then*

(i) If (Γ, P) also satisfies the condition in Theorem 1.1 (i), then (Γ, P) has quadratic mixing time, i.e. for $1 \leq p \leq \infty$,

$$\tau_p(\epsilon) = \mathcal{O}_\delta\left(\frac{\gamma^2}{\eta}\log \mathcal{O}_\delta\left(\frac{1}{\epsilon}\right)\right)$$

(ii) For a lower bound, if (Γ, P) satisfies the condition in Theorem 1.1 (ii), we have

$$\tau_p(\epsilon) \geq C(\delta, \epsilon)\gamma^2$$

where $C(\delta,\epsilon) = \frac{1}{2^{O_\delta(1)}O_\delta(1)} \log\frac{1}{\epsilon} = \frac{1}{O_\delta(1)}\log\frac{1}{\epsilon}$.

Remark. At the moment, we are unable to make the bound in Corollary 7.2 explicit. This is due to our reliance on the non-explicit Theorem7.1. In particular, we do not know how the bound for the mixing time depends on δ. For a deeper discussion about this, we refer the reader to the remark in [4, p. 4]

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