RESEARCH INTERESTS

My research lies in the area of geometric group theory. Broadly speaking, I enjoy leveraging techniques and concepts from other areas of mathematics to progress in group theory. In my work to date, I employed geometric, number-theoretic, probabilistic, and combinatorial tools to explore algebraic structures of infinite groups. More specifically, I have explored topics such as the study of infinite groups through their finite approximations, the growth of groups, subgroup distortion in solvable groups, the conjugacy ratio and conjugacy growth in finitely generated groups.

Detailed Research Statement

***Theme 1.1:* Residually finite groups with uniformly almost flat quotients** (Ongoing work with Dr Sean Eberhard, University of Warwick)
Given a residually finite group, there has been increasing effort in understanding what information is encoded in the finite quotients. Khukhro and Valette examined a geometric property of these finite quotients known as the *diameter*. They investigated which algebraic properties can be deduced from the diameter of finite quotients. They showed that the finite quotients have diameter bounded uniformly below by a constant multiple of their size if and only if the group is virtually cyclic. In addition, using the work of Breuillard and Tointon, they showed that the diameter of a certain sequence of finite quotients is bounded uniformly below by a polynomial in their size if and only if the group virtually maps onto $\mathbb{Z}$. Both of these are intriguing results which link an easily understood algebraic property with a seemingly unrelated geometric property.

Tointon and I proved some significant generalisations of their results [7]. To be precise, let G be a group generated by a finite symmetric subset S containing the identity. Given a subgroup $H \leq G$, define $\mathrm{diam}_S(G/H) = \min\{n \in \mathbb{N} : S^n H = G\}$. Given $\alpha \in (0,1]$, we say that a residually finite group G has *uniformly α-almost flat quotients (resp. coset spaces)* if there exists $\varepsilon > 0$ such that for every normal subgroup $H \trianglelefteq G$ (resp. subgroup $H \leqslant G$) of finite index we have $\mathrm{diam}_S(G/H) \geq \varepsilon[G:H]^\alpha$. We proved that a polycyclic group has uniformly almost flat coset spaces if and only if it is virtually nilpotent; also, a residually torsion-free nilpotent group has uniformly almost flat quotients if and only if it is virtually nilpotent. Encouraged by these results, we take this opportunity to propose future research to characterise finitely generated (f.g.) residually finite groups that satisfy this condition.

Problem 1. *Characterise algebraically those f.g. residually finite groups that have uniformly almost flat coset spaces or uniformly almost flat quotients.*

Answering either aspect of Conjecture 1 seems ambitious. In particular, the proof of the polycyclic case is much harder than the proof of Wolf's Theorem (e.g. see [5, Theorem 14.30]), for example relying on a deep result by Breuillard, Green and Tao on the structure of approximate groups.

As a starting point, I would like to, for as broad a class of groups as possible, establish the equivalence of having uniformly almost flat coset spaces and being virtually nilpotent via the similar strategy of Gromov's theorem. Most proofs of Gromov's theorem (e.g. see [9]) involve two key separate steps:

(1) Reducing the general case to the polycyclic case.
(2) Applying Wolf's theorem which deals with the special case of the polycyclic group.

Since Tointon and I have solved the coset aspect of Conjecture 1 for the polycyclic groups, I plan to reduce Conjecture 1 to our result for any hereditary class of f.g. groups satisfying the following property.

Problem 2. *For which f.g. infinite residually finite group G does the following statement hold? Let $d \in \mathbb{N}$. Suppose G has uniformly $(1/d)$-almost flat quotients, and H is a finite-index subgroup admitting a quotient $H/K \cong \mathbb{Z}$. Then K is f.g. and has uniformly $(1/(d-1))$-almost flat quotients.*

Indeed, if we solve Conjecture 2, we can use Khukhro and Valette's result [8] and induction on d, to show that having uniformly almost flat coset spaces implies the group is virtually polycyclic. I have taken the initial step and solved the first part of Conjecture 2 when K is nilpotent (not necessarily f.g.).

***Theme 2.1:* The conjugacy ratio of f.g. groups**
We now discuss a property of f.g. groups introduced by Ciobanu, Cox, and Martino (CCM), called the *conjugacy ratio*. In a finite group, this ratio, the number of conjugacy classes divided by the group order, coincides with the degree of commutativity, the probability that two elements commute, and both notions have been generalised to infinite groups, where their behaviour reflects algebraic, probabilistic, and geometric aspects. Let $B(r)$ be the ball of radius r with respect to a generating set S, and let $c(r)$ be the number of distinct conjugacy classes intersecting $B(r)$. The conjugacy ratio is defined as $\mathrm{cr}_S(G) = \limsup_{r\to\infty} c(r)/|B(r)|$. CCM conjectured that $\mathrm{cr}_S(G) > 0$ if and only if G is virtually abelian; they verified this for groups with polynomial growth and suggested studying groups with exponential growth.

Conjecture 3. *The conjugacy ratio of a f.g. soluble group is positive if and only if it is virtually abelian.*

I have successfully verified Conjecture 3 for the abelian-by-cyclic case [6]. I also discussed with Alex Evetts, who works on conjugacy geodesics of nilpotent groups, and we developed approaches to extend this result. My research for Conjecture 3 led me to another key area of geometric group theory: *subgroup distortion*. In general, describing geodesics in arbitrary groups is difficult, making the conjugacy ratio, defined via the size of balls and number of intersecting conjugacy classes, seem challenging to compute. However, in abelian-by-cyclic groups, I found that techniques from subgroup distortion allow us to bypass computing complete geodesic sets. The distortion of a subgroup measures how parts of the group are stretched or twisted. Using this insight, I significantly simplified the computation of conjugacy ratio for infinite groups, in particular by examining subgroups with so-called polynomial distortion. This naturally leads to the problem of classifying all subgroups with polynomial distortion in a given group.

Theme 2.2: ***Subgroup distortion in Soluble groups*** Again, as a starting point, I would like to consider the following case:

Problem 4. *Classifying all subgroups with polynomial distortion in soluble groups.*

I have discussed this problem with Mark Hagen, an expert in geometric group theory. I have made some substantial progress in my conjugacy ratio project where I look at the basic case of abelian-by-cyclic groups. This would go significantly beyond the work of Davis and Olshanskii [4], who answered this question for wreath products of f.g. abelian with $\mathbb{Z}$. Again, a plausible approach for Conjecture 4 is to extend the abelian-by-cyclic to the general polycyclic case.

The essential concept for both Conjecture 3 and Conjecture 4 is understanding geodesics in soluble groups. My approach is to extend the progress made in the abelian-by-cyclic case to the general soluble case, analogous to Milnor–Wolf's theorem. In their proof, they first analysed $\mathbb{Z}^n \rtimes \mathbb{Z}$, then nilpotent-by-$\mathbb{Z}$, then polycyclic, and finally general soluble groups. We adopted a similar strategy in our results for polycyclic groups in Theme 1.1. As an initial step, consider the metabelian group $G = \mathbb{Z}^n \rtimes_\Phi \mathbb{Z}^m$ with the standard generating set. For $m = 1$, Parry [10, Section 3] showed that an arbitrary element can be represented as a geodesic:

$$g = v_0 u_1 v_1 u_2 v_2 \dots u_{h-1} v_{h-1} u_h v_h, \tag{1}$$

where $h \geq 0$, v_i represent elements in $\mathbb{Z}^m$, and u_i represent elements in $\mathbb{Z}^n$. I aim to extend this description for arbitrary $m \in \mathbb{N}$. In particular, for $n \geq 2$, if most elements of G can be expressed in this form, then the subgroup P of periodic points of Φ in $\mathbb{Z}^n$ is polynomially distorted in G, which is a crucial step toward solving Conjecture 4 for metabelian groups. Combined with Breuillard and Cornulier's result that every f.g. solvable group with exponential growth has exponential conjugacy growth [1], this provides a foundation for Conjecture 3.

Theme 2.3: ***Conjugacy growth in abelian-by-cyclic groups*** (Ongoing work with Prof. Laura Ciobanu, TU Berlin, and Dr Alex Evetts, Manchester)

Consider $BS(1,k)$ with the standard generating set and let $c(r)$ count conjugacy classes intersecting the ball of radius r. In [2], Ciobanu, Evetts, and Ho showed $c(r) \sim \frac{\alpha^r}{r}$ for $\alpha > 1$, similar to hyperbolic and some acylindrically hyperbolic groups, despite $BS(1,k)$ not being acylindrically hyperbolic. We aim to extend this to abelian-by-cyclic groups using the idea in my paper [6].

Conjecture 5. *The conjugacy growth $c(r)$ of an abelian-by-cyclic group with exponential growth satisfies $c(r) \sim \frac{\beta^r}{r}$ for some $\beta > 1$.*

Sketch of approach: Let $G = K \rtimes_\phi \langle t \rangle$ be abelian-by-cyclic with exponential growth. We need to verify asymptotically:

(1) The growth of G is α^r for some $\alpha > 1$. (not true for general groups with exponential growth, e.g. consider $F_2 \oplus F_2$, the direct product of free groups with respect to the standard generating set)

(2) The number of elements with zero t-exponential sum is negligible in the ball of radius r.
(3) The cyclic permutation of a geodesic with non-zero t-exponential produces "many" distinct elements, and most conjugating elements on the sphere are of this form.

***Theme 2.4:* Twisted conjugacy growth in abelian-by-cyclic groups** (Ongoing project with Dr Alex Evetts, Manchester)

Twisted conjugacy arises in Reidemeister theory, Selberg theory, and algebraic geometry. Moreover, conjugacy in extensions can be understood via twisted conjugacy in the base group.

Let G be f.g. with automorphism φ. Elements $x, y \in G$ are φ-twisted conjugate if $\exists z \in G$ with $zx\varphi(z)^{-1} = y$. The equivalence classes $[x]_\varphi$ are the φ-twisted conjugacy classes.

Though studied, twisted conjugacy in abelian-by-cyclic groups outside RAAGs is limited [3]. It is clear that the conjugacy growth function is bounded above by the standard growth function. In fact, there exists a finite greatened infinite group with just 2 conjugacy classes. However, can similar phenomena occur with twisted conjugacy growth? More specifically, can the growth of twisted conjugacy growth be slower than the conjugacy ratio growth? Building on my experience with automorphisms (mentioned in Theme 1.1) and conjugacy geodesic (mentioned in Theme 2.1) in abelian-by-cyclic groups, Alex and I came up with a concrete question which I will study during my fellowship:

Problem 6. *Consider the group $BS(1,k)$, whose conjugacy growth (hence the standard growth) is exponential. Does there exist an automorphism ϕ of $BS(1,k)$ such that the corresponding ϕ-twisted conjugacy growth function is slower than exponential?*

***Theme 2.5:* Degree of commutativity and conjugacy ratio in acylindrically hyperbolic groups** (Ongoing project with Dr Abdul Zalloum, Queens University, Canada)

Problem 7. *Every acylindrically hyperbolic group has exponential conjugacy growth. Using the fact that such groups contain many free subgroups, we aim to show that both their degree of commutativity and conjugacy ratio are* 0.

References

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